Sira Gratz — <u>The spectrum of a graded comm. alg.</u> — Talk 1, Day 2 —

- **Def** A ring R is <u>graded</u> if $R = \bigoplus_{i\in\mathbb{Z}} R_i$ s.t. $R_iR_j \subseteq R_{i+j}$.
  $r \in R_i$ <u>homog.</u> of degree i : $|r| = i$.

<u>Ex</u>: $k[x]$, $|x| = 1$

Throughout, let R be a (commutative (graded ring)).

<u>Rmk</u>: Weaker notion of <u>graded commutativity</u>: $r\cdot s = (-1)^{|r|\cdot|s|} s\cdot r$ for r,s homogeneous.

Reason: everything we do is controlled by the even part.

$$R_{ev} := \bigoplus_{i\in\mathbb{Z}} R_{2i}$$

Def: An R-module M is a <u>graded R-module</u> if $M = \bigoplus_{i\in\mathbb{Z}} M_i$ s.t.

$$R_i \cdot M_j \subseteq M_{i+j}$$

- <u>Familiar example</u>: gp. cohom. $H^*(G,k) := \operatorname{Ext}^*_{kG}(k,k) = \bigoplus_{i\geq 0} \operatorname{Ext}^i_{kG}(k,k)$
  is a graded comm., <u>graded connected</u> k-algebra:
  concentrated in non-neg. degrees and $R_0 \cong k$ (the base field).

If $M \in \operatorname{mod} kG$, then $\bigoplus_i \operatorname{Ext}^i_{kG}(k,M)$ is a graded $\operatorname{Ext}^*(k,k)$-module.

Def: <u>$\operatorname{grmod} R$</u> := category of f.g. graded R-modules
with degree-zero maps: $f : \bigoplus_i M_i \to \bigoplus_j N_j$ R-lin
& s.t. $f(M_i) \subseteq N_i$.

Def: tensor product $(M \otimes_R N)_n := \left(\bigoplus_{i+j=n} M_i \otimes_k N_j \right) \Big/ \left(mr\otimes n - m \otimes rn \;\middle|\; |m| = p,\ |n| = j,\ |r| = i-p \right)$
(working over a field k)

$-\otimes_R-$ makes $\operatorname{grmod} R$ symm. monoidal.

- Graded ideals & graded prime ideals

Def: a <u>graded ideal</u> is a graded sub-mod of R ($\Leftrightarrow$ ideal gen. by homog. elements)

Def: A gr noeth. ring R is <u>graded noeth.</u> if it satisfies the ACC for graded ideals.

Lemma $\Leftrightarrow$ … $\Leftrightarrow$ R_0 is noeth. & R is f.g. R_0-alg.

<u>Examples</u>: • $k[x_1,\dots,x_n]$, $|x_i| = m_i$, is noeth. (easy).

• $H^*(G;k)$ is noeth. [Evens, Venkov]

<u>Point</u>:
If R is gr. noeth. $\Rightarrow$ grmod R is abelian.

<u>Def</u>: A graded ideal $\mathfrak{p}$ is graded prime if $r \cdot s \in \mathfrak{p}$ (r,s homog.) $\Rightarrow$ $r \in \mathfrak{p}$ or $s \in \mathfrak{p}$.

• <u>Def</u>: $\mathrm{Spec}^h(R) := \{ \mathfrak{p} \subsetneq R \mid \mathfrak{p} \text{ graded commutative} \}$
The <u>homog. spectrum</u> of R.

Zariski topology (gr. version): basis of closed subsets given by: $\{ V(I) := \{ \mathfrak{p} \in \mathrm{Spec}^h R \mid \mathfrak{p} \supseteq I \} \}_{I \subseteq R \text{ gr. ideal}}$.

• <u>Example</u>: $k[x]$, $|x| = 1$ $\rightsquigarrow$ $\mathrm{Spec}^h k[x] = \{ (0), (x) \}$.

<u>Recall</u>, we want: $D^b(kG) \supseteq$ thick $\otimes$-ideals $\overset{1-1}{\longleftrightarrow}$ spc-cl subsets $\subseteq \mathrm{Spec}^h(H^*(G,k))$
mod kG $\quad \dots \quad \overset{1-1}{\longleftrightarrow} \quad \dots \quad$ Proj (——)

Spec^h: $\mathrm{CGrRings}^{op} \longrightarrow \mathrm{Top}$ contravariant functor:
$(f: R \to S) \longmapsto (\mathrm{Spec}^h(f): \mathfrak{q} \mapsto f^{-1}\mathfrak{q})$

• If R is non-neg. graded, then $R_{\geq 1} := \bigoplus_{i>0} R_i$ is a graded ideal. "irrelevant prime"

• <u>Def</u>: $\mathrm{Proj}\, R := \mathrm{Spec}^h R \setminus V(R_{\geq 1})$, open subspace of $\mathrm{Spec}^h R$.

<u>Example</u>: $\mathrm{Proj}\, \underbrace{k[x_0,\dots,x_n]}_{R}$, $|x_0| = \dots = |x_n| = 1$

$\rightsquigarrow$ $R_{\geq 1} = (x_0,\dots,x_n) = V(R_{\geq 1})$ (one point!) is the space underlying $\mathbb{P}^n_k$.

Quotients & localizations

- $I \subseteq R$ graded id $\Rightarrow$ R/I is a naturally gr. ring s.t.

  $\pi : R \to R/I$ is a degree 0 map. $\left(R/I = \bigoplus_{i \in \mathbb{Z}} R_i / I_i \right)$

<u>Lemma</u>: $\pi : R \to R/I$ induces a closed embedding

$\mathrm{Spec}^h (R/I) \hookrightarrow \mathrm{Spec}^h (R)$ with image $V(I)$.

- $S \subseteq R$ multiplicative subset of homog. elts

  $\Rightarrow$ $S^{-1}R$ is naturally graded: $(S^{-1}R)_i = \{ \frac{r}{s} \mid |r| - |s| = i \}$.

<u>Example</u>:
- $R = k[x]$, $|x| = 1$, $S := \{ x^n \mid n \geq 0 \}$

  $\Rightarrow$ $S^{-1}R = k[x, x^{-1}]$. $|x^{-1}| = -1$.

- $\mathfrak{p} \in \mathrm{Spec}^h R$, $S = \{ r \in \bigcup_{i \in \mathbb{Z}} R_i \mid r \notin \mathfrak{p} \}$ $\Rightarrow$ $R_{\mathfrak{p}} := S^{-1}R$.

(Previous example: $k[x, x^{-1}] = R_{(0)}$, loc. at the prime (0)).

<u>Lemma</u>: The map $\begin{array}{rcl} R & \longrightarrow & S^{-1}R \\ r & \longmapsto & \frac{r}{1} \end{array}$ induces an injective map

$\mathrm{Spec}^h S^{-1}R \hookrightarrow \mathrm{Spec}^h R$ with image:

$\{ \mathfrak{p} \in \mathrm{Spec}^h R \mid \mathfrak{p} \cap S = \emptyset \}$.

Local rings & residue fields

<u>Def</u>. R is <u>graded local</u> if it ~~meets~~ admits a unique maximal ideal.

<u>Ex</u>:
- If $\mathfrak{p} \in \mathrm{Spec}^h R$, then $R_{\mathfrak{p}}$ is gr-local with max. id $\mathfrak{p} \cdot R_{\mathfrak{p}}$.
- If R is gr-connected, then R is graded local & the max. ideal is the irrelevant ideal $R_{\geq 1}$.

Lemma (Graded Nakayama)

If R is graded local with maximal ideal $\mathfrak{m}$, M a f.g. graded R-module. If $\mathfrak{m} \cdot M = M \Rightarrow M = 0$.

Pf: If R is gr. connected k-algebra, M f.g. If $M \neq 0$,

$\Rightarrow \exists\, i \in \mathbb{Z}$ s.t. $M_i \neq 0$ but $M_{<i} = 0$.

If $\underbrace{\mathfrak{m}}_{= R_{\geq 1}} \cdot M = M \Rightarrow M_i = (R_{\geq 1} \cdot M)_i = 0$ ↯.

□ (gr. conn. case only)

Def: A graded ring R is a <u>graded field</u> if all homog. elements $\neq 0$ are invertible.

Ex: • R gr. local & max ideal $\mathfrak{m} \Rightarrow R/\mathfrak{m}$ is a graded field, called the <u>graded residue field</u>.

• $\mathfrak{p} \in \operatorname{Spec}^h R$, then <u>the res. field at $\mathfrak{p}$</u> is $k(\mathfrak{p}) := R_{\mathfrak{p}} / \mathfrak{p} \cdot R_{\mathfrak{p}}$.

• Concrete: $k[x]$, $|x| = 1$: $k(0) = k[x]_{(0)} = k[x, x^{-1}]$
a graded field but <u>not</u> a field ($x-1$ e.g. not inv'ble!).

Lemma: Let K be a gr. field. Then every graded K-module is graded free: $M = \bigoplus_i K(n_i)$

"shifted" module: the K-module with $K(n_i)_j = K_{j+n_i}$.

Corollary (of Nakayama)

If R is gr. local with res. field k, $0 \neq M$ a f.g. R-module. Then:

i) $M \otimes_R k \neq 0$

ii) M admits a minimal set of gen.

iii) M admits a graded free cover.

$M \in \mathrm{grmod}\, R$. <u>Def</u>: $\underset{\sim\sim}{\mathrm{Supp}_R}\, M := \{ \mathfrak{p} \in \mathrm{Spec}^h R \mid M_{\mathfrak{p}} \neq 0 \}$

where $M_{\mathfrak{p}} := M \otimes_R R_{\mathfrak{p}}$.

<u>Lemma</u>: $\mathrm{Supp}_R M = \emptyset \iff M \cong 0$.

<u>Lemma</u>: M f.g., then $\mathrm{Supp}_R M$ is closed, in fact $\mathrm{Supp}_R M = V(\mathrm{Ann}_R M)$.

<u>Lemma</u>: If M f.g., then $\mathrm{Supp}_R M = \{ \mathfrak{p} \in \mathrm{Spec}^h R \mid M \otimes_R k(\mathfrak{p}) \neq 0 \}$

<u>Pf</u>: $\mathfrak{p} \in \mathrm{Supp}_R M \iff M_{\mathfrak{p}} \neq 0$, a f.g. $R_{\mathfrak{p}}$-module (inherited from M over R)

By Nakayama: $\iff \underbrace{(\mathfrak{p} \cdot R_{\mathfrak{p}})}_{\text{the max. ideal in } R_{\mathfrak{p}}} \cdot M_{\mathfrak{p}} \neq M_{\mathfrak{p}} \iff M_{\mathfrak{p}} / (\mathfrak{p} R_{\mathfrak{p}}) M_{\mathfrak{p}} \neq 0$

$$M_{\mathfrak{p}} / (\mathfrak{p} R_{\mathfrak{p}}) M_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} k(\mathfrak{p}) \cong M \otimes_R k(\mathfrak{p})$$

$\square$
